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Strong convergence theorems for solving a general system of finite variational inequalities for finite accretive operators and fixed points of nonexpansive semigroups with weak contraction mappings

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Abstract
In this paper, we prove a strong convergence theorem for finding a common solution of a general system of finite variational inequalities for finite different inverse-strongly accretive operators and solutions of fixed point problems for a nonexpansive semigroup in a Banach space based on a viscosity approximation method by using weak contraction mappings. Moreover, we can apply the above results to find the solutions of the class of *k*-strictly pseudocontractive mappings and apply a general system of finite variational inequalities into a Hilbert space. The results presented in this paper extend and improve the corresponding results of Ceng *et al.* (2008), Katchang and Kumam (2011), Wangkeeree and Preechasilp (2012), Yao *et al.* (2010) and many other authors.
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1 Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. Let E^* be the dual space of E and $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by $J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$ for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and, usually, write $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q: (i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$; (ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$; and (iii) $J_q(-x) = -J_q(x)$ for all $x \in E$. It is known that if E is smooth, then J is single-valued, which is denoted by j. Recall that the duality mapping j is said to be weakly sequentially continuous if for each $x_n \to x$ weakly, we have $j(x_n) \to j(x)$ weakly-*. We know that if E admits a weakly sequentially continuous duality mapping, then E is smooth (for the details, see [24, 25, 29]).

Let $f : C \to C$ be a *k-contraction mapping* if there exists $k \in [0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x, y \in C$. Let $S : C \to C$ a nonlinear mapping. We use $F(S)$ to denote the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping S is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. A mapping f is called *weakly contractive* on a closed convex set C in the Banach space E if there exists $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$, $\lim_{t \to \infty} \varphi(t) = \infty$ and $x, y \in C$

$$\left\| f(x) - f(y) \right\| \leq \|x - y\| - \varphi\left(\|x - y\|\right). \tag{1.1}$$

If $\varphi(t) = (1 - k)t$, then f is called to be *contractive* with the contractive coefficient k. If $\varphi(t) \equiv 0$, then f is said to be *nonexpansive*.

A family $\mathcal{S} = \{T(t) : t \geq 0\}$ of mappings of C into itself is called a *nonexpansive semigroup* (see also [14]) on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common *fixed points of* $\mathcal{S}$, that is,

$$F(\mathcal{S}) = \bigcap_{t=0}^{\infty} F\bigl(T(t)\bigr) = \bigl\{x \in C : T(t)x = x, 0 \leq t < \infty\bigr\}.$$

It is known that $F(\mathcal{S})$ is closed and convex. Moreover, for the study of nonexpansive semigroup mapping, see [5, 14–16, 26] for more details.

In 2002, Suzuki [21] was the first one to introduce the following implicit iteration process in Hilbert spaces:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)(x_n), \quad n \geq 1 \tag{1.2}$$

for the nonexpansive semigroup. In 2007, Xu [28] established a Banach space version of the sequence (1.2) of Suzuki [21]. In [4], Chen and He considered the viscosity approximation process for a nonexpansive semigroup and proved another strong convergence theorems for a nonexpansive semigroup in Banach spaces, which is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $f : C \to C$ is a fixed contractive mapping. Recall that an operator $A : C \to E$ is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\bigl\langle Ax - Ay, j(x - y)\bigr\rangle \geq 0$$

for all $x, y \in C$. A mapping $A : C \to E$ is said to be β*-strongly accretive* if there exists a constant $\beta > 0$ such that

$$\bigl\langle Ax - Ay, j(x - y)\bigr\rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C.$$

An operator $A : C \to E$ is said to be β*-inverse strongly accretive* if, for any $\beta > 0$,

$$\bigl\langle Ax - Ay, j(x - y)\bigr\rangle \geq \beta \|Ax - Ay\|^2$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator. To convey an idea of the *variational inequality*, let C be a closed and convex set in a real Hilbert space H. For a given operator A, we consider the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0$$

for all $x \in C$, which is known as the variational inequality, introduced and studied by Stampacchia [22] in 1964 in the field of potential theory. In 2006, Aoyama *et al.* [1] first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, j(y - x) \rangle \geq 0 \tag{1.4}$$

for all $y \in C$. This problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [10, 11]. In order to find a solution of the variational inequality (1.4), Aoyama *et al.* [1] proved the strong convergence theorem in the framework of Banach spaces which is generalized by Iiduka *et al.* [8] from Hilbert spaces.

Motivated by Aoyama *et al.* [1] and also Ceng *et al.* [3], Qin *et al.* [18] and Yao *et al.* [29] first considered the following *new general system of variational inequalities* in Banach spaces:

Let $A : C \to E$ be a β-inverse strongly accretive mapping. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.5}$$

Let C be nonempty closed convex subset of a real Banach space E. For two given operators $A, B : C \to E$, consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.6}$$

where λ and μ are two positive real numbers. This system is called the *general system of variational inequalities* in a real Banach spaces. If we add up the requirement that $A = B$, then the problem (1.6) is reduced to the system (1.5).

By the following general system of variational inequalities, we extend into the *general system of finite variational inequalities* which is to find $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ and is defined by

$$\begin{cases} \langle \lambda_M A_M x_M^* + x_1^* - x_M^*, j(x - x_1^*) \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, j(x - x_M^*) \rangle \geq 0, & \forall x \in C, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, j(x - x_3^*) \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, j(x - x_2^*) \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

where $\{A_l\}_{l=1}^{M}: C \to E$ is a family of mappings, $\lambda_l \geq 0$, $l \in \{1, 2, \dots, M\}$. The set of solutions of (1.7) is denoted by $\mathrm{GSVI}(C, A_l)$. In particular, if $M = 2$, $A_1 = B$, $A_2 = A$, $\lambda_1 = \mu$, $\lambda_2 = \lambda$, $x_1^* = x^*$ and $x_2^* = y^*$, then the problem (1.7) is reduced to the problem (1.6).

In this paper, motivated and inspired by the idea of Ceng *et al.* [3], Katchang and Kumam [12] and Yao *et al.* [29], we introduce a new iterative scheme with weak contraction for finding solutions of a new general system of finite variational inequalities (1.7) for finite different inverse-strongly accretive operators and solutions of fixed point problems for nonexpansive semigroups in a Banach space. Consequently, we obtain new strong convergence theorems for fixed point problems which solve the general system of variational inequalities (1.6). Moreover, we can apply the above theorem to finding solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of Ceng *et al.* [3], Katchang and Kumam [12], Wangkeeree and Preechasilp [26], Yao *et al.* [29] and many other authors.

2 Preliminaries

We always assume that E is a real Banach space and C is a nonempty closed convex subset of E.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}\big(\|x + y\| + \|x - y\|\big) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$: see, for instance, [1, 24].

We note that E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$; see [24, 27] for more details.

Let D be a subset of C and $Q : C \to D$. Then Q is said to be *sunny* if

$$Q\big(Qx + t(x - Qx)\big) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z in the range of Q. For example, see [1, 23] for more details. The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1 ([19]) *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in E$;
(iii) $\langle x - Qx, J(y - Qx)\rangle \le 0$, $\forall x \in E$, $y \in C$.

Proposition 2.2 ([13]) *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with $F(T) \ne \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } x \ne y.$$

By [7, Theorem 1], it is well known that, if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition and E is smooth.

We need the following lemmas for proving our main results.

Lemma 2.3 ([27]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.4 ([20]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5 (Lemma 2.2 in [17]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real number sequences and $\{\alpha_n\}$ a positive real number sequence satisfying the conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{\alpha_n} = 0$. Let the recursive inequality*

$$a_{n+1} \le a_n - \alpha_n \varphi(a_n) + b_n, \quad n \ge 0,$$

where $\varphi(a)$ is a continuous and strict increasing function for all $a \ge 0$ with $\varphi(0) = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([6]) *Let E be a uniformly convex Banach space and $B_r(0) := \{x \in E : \|x\| \le r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g\big(\|x - y\|\big)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.7 ([2]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.8 (Yao *et al.* [29, Lemma 3.1]; see also [1, Lemma 2.8]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be β-inverse-strongly accretive. Then, we have*

$$\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2 \leq \|x-y\|^2 + 2\lambda\left(\lambda K^2-\beta\right)\|Ax-Ay\|^2.$$

If $\beta \geq \lambda K^2$, then $I-\lambda A$ is nonexpansive.

3 Main results

In this section, we prove a strong convergence theorem. In order to prove our main results, we need the following two lemmas.

Lemma 3.1 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let the mapping $A_l : C \to E$ be a β_l-inverse-strongly accretive such that $\beta_l \geq \lambda_l K^2$ where $l \in \{1,2,\dots,M\}$. If $\mathcal{Q} : C \to C$ is a mapping defined by*

$$\mathcal{Q}(x) = Q_C(I-\lambda_M A_M)Q_C(I-\lambda_{M-1}A_{M-1})\cdots Q_C(I-\lambda_2 A_2)Q_C(I-\lambda_1 A_1)x, \quad \forall x \in C,$$

then $\mathcal{Q}$ is nonexpansive.

Proof Taking $\mathcal{Q}_C^l = Q_C(I-\lambda_l A_l)Q_C(I-\lambda_{l-1}A_{l-1})\cdots Q_C(I-\lambda_2 A_2)Q_C(I-\lambda_1 A_1)$, $l \in \{1,2,3,\dots,M\}$ and $\mathcal{Q}_C^0 = I$, where I is the identity mapping on E, we have $\mathcal{Q} = \mathcal{Q}_C^M$. For any $x, y \in C$, we have

$$\begin{aligned}
\left\|\mathcal{Q}(x)-\mathcal{Q}(y)\right\| &= \left\|\mathcal{Q}_C^M x - \mathcal{Q}_C^M y\right\| \\
&= \left\|Q_C(I-\lambda_M A_M)\mathcal{Q}_C^{M-1}x - Q_C(I-\lambda_M A_M)\mathcal{Q}_C^{M-1}y\right\| \\
&\leq \left\|(I-\lambda_M A_M)\mathcal{Q}_C^{M-1}x - (I-\lambda_M A_M)\mathcal{Q}_C^{M-1}y\right\| \\
&\leq \left\|\mathcal{Q}_C^{M-1}x - \mathcal{Q}_C^{M-1}y\right\| \\
&\vdots \\
&\leq \left\|\mathcal{Q}_C^0 x - \mathcal{Q}_C^0 y\right\| \\
&= \|x-y\|.
\end{aligned}$$

Therefore, $\mathcal{Q}$ is nonexpansive. □

Lemma 3.2 *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A_l : C \to E$ be nonlinear mapping, where $l \in \{1,2,\dots,M\}$. For $x_l^* \in C$, $l \in \{1,2,\dots,M\}$, $(x_1^*, x_2^*, \dots, x_M^*)$ is a solution of* *problem (1.7) if and only if*

$$\begin{cases} x_1^* = Q_C(I - \lambda_M A_M)x_M^*, \\ x_2^* = Q_C(I - \lambda_1 A_1)x_1^*, \\ x_3^* = Q_C(I - \lambda_2 A_2)x_2^*, \\ \vdots \\ x_M^* = Q_C(I - \lambda_{M-1} A_{M-1})x_{M-1}^*, \end{cases} \tag{3.1}$$

that is

$$x_1^* = Q_C(I - \lambda_M A_M)Q_C(I - \lambda_{M-1}A_{M-1}) \cdots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x_1^*.$$

Proof From (1.7), we rewrite as

$$\begin{cases} \langle x_1^* - (x_M^* - \lambda_M A_M x_M^*), j(x - x_1^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle x_M^* - (x_{M-1}^* - \lambda_{M-1} A_{M-1} x_{M-1}^*), j(x - x_M^*)\rangle \geq 0, \quad \forall x \in C, \\ \vdots \\ \langle x_3^* - (x_2^* - \lambda_2 A_2 x_2^*), j(x - x_3^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle x_2^* - (x_1^* - \lambda_1 A_1 x_1^*), j(x - x_2^*)\rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{3.2}$$

Using Proposition 2.1(iii), the system (3.2) is equivalent to (3.1). □

Throughout this paper, the set of fixed points of the mapping $\mathcal{Q}$ is denoted by $F(\mathcal{Q})$. The next result states the main result of this work.

Theorem 3.3 *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C and Q_C be a sunny nonexpansive retraction from E onto C. Let $A_l : C \to E$ be a β_l-inverse-strongly accretive such that $\beta_l \geq \lambda_l K^2$, where $l \in \{1, 2, \dots, M\}$, and K be the best smooth constant. Let f be a weakly contractive mapping on C into itself with function φ. Suppose $\mathcal{F} := F(\mathcal{Q}) \cap F(\mathcal{S}) \neq \emptyset$, where $\mathcal{Q}$ is defined by Lemma 3.1. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = Q_C(I - \lambda_M A_M)Q_C(I - \lambda_{M-1}A_{M-1}) \cdots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n, \end{cases} \tag{3.3}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$ and satisfy $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1$, $n \geq 1$, $\{\mu_n\} \subset (0, \infty)$, and λ_l, $l = 1, 2, \dots, M$ are positive real numbers. The following conditions are satisfied:

- *(C1)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- *(C2)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- *(C3)* $\lim_{n\to\infty} \mu_n = 0$;
- *(C4)* $\lim_{n\to\infty} \sup_{y \in \tilde{C}} \|T(\mu_{n+1})y - T(\mu_n)y\| = 0$, *$\tilde{C}$ bounded subset of C.*

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$ and $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_M)$ is a solution of the problem (1.7) where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof First, we prove that $\{x_n\}$ is bounded. Let $p \in \mathcal{F}$, taking

$$\mathcal{Q}_C^l = Q_C(I - \lambda_l A_l) Q_C(I - \lambda_{l-1} A_{l-1}) \cdots Q_C(I - \lambda_2 A_2) Q_C(I - \lambda_1 A_1), \quad l \in \{1, 2, 3, \ldots, M\},$$

$\mathcal{Q}_C^0 = I$, where I is the identity mapping on E. From the definition of Q_C is nonexpansive then $\mathcal{Q}_C^l$, $l \in \{1, 2, 3, \ldots, M\}$ also. We note that

$$\|y_n - p\| = \left\| \mathcal{Q}_C^l x_n - \mathcal{Q}_C^l p \right\| \le \|x_n - p\|. \tag{3.4}$$

From (3.3) and (3.4), we also have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) y_n - p \right\| \\
&\le \alpha_n \left\| f(x_n) - p \right\| + \beta_n \|x_n - p\| + \gamma_n \left\| T(\mu_n) y_n - T(\mu_n) p \right\| \\
&\le \alpha_n \left[\|x_n - p\| - \varphi\left(\|x_n - p\|\right) \right] + \alpha_n \left\| f(p) - p \right\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\
&\le \|x_n - p\| - \alpha_n \varphi\left(\|x_n - p\|\right) + \alpha_n \left\| f(p) - p \right\| \\
&\le \max\left\{ \|x_1 - p\|, \varphi\left(\|x_1 - p\|\right), \left\| f(p) - p \right\| \right\}.
\end{aligned} \tag{3.5}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, and $\{T(\mu_n) y_n\}$.

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Notice that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| \mathcal{Q}_C^M x_{n+1} - \mathcal{Q}_C^M x_n \right\| \\
&= \left\| Q_C(I - \lambda_M A_M) \mathcal{Q}_C^{M-1} x_{n+1} - Q_C(I - \lambda_M A_M) \mathcal{Q}_C^{M-1} x_n \right\| \\
&\le \left\| (I - \lambda_M A_M) \mathcal{Q}_C^{M-1} x_{n+1} - (I - \lambda_M A_M) \mathcal{Q}_C^{M-1} x_n \right\| \\
&\le \left\| \mathcal{Q}_C^{M-1} x_{n+1} - \mathcal{Q}_C^{M-1} x_n \right\| \\
&\ \ \vdots \\
&\le \left\| \mathcal{Q}_C^0 x_{n+1} - \mathcal{Q}_C^0 x_n \right\| \\
&= \|x_{n+1} - x_n\|.
\end{aligned}$$

Setting $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$ for all $n \ge 0$, we see that $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. Then we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T(\mu_{n+1}) y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n T(\mu_n) y_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T(\mu_{n+1}) y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n+1} f(x_n)}{1 - \beta_{n+1}} + \frac{\alpha_{n+1} f(x_n)}{1 - \beta_{n+1}} \right. \\
&\quad \left. - \frac{\gamma_{n+1} T(\mu_n) y_n}{1 - \beta_{n+1}} + \frac{\gamma_{n+1} T(\mu_n) y_n}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n T(\mu_n) y_n}{1 - \beta_n} \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1}) - f(x_n)\big) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\big(T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\big) \right. \\
&\quad \left. + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right) f(x_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) T(\mu_n)y_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(x_{n+1}) - f(x_n)\big\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\big\|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\big\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|f(x_n)\big\| + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right| \big\|T(\mu_n)y_n\big\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(x_{n+1}) - f(x_n)\big\| \\
&\quad + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}\big(\|y_{n+1} - y_n\| + \big\|T(\mu_{n+1})y_n - T(\mu_n)y_n\big\|\big) \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|f(x_n)\big\| + \left|\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} - \frac{1-\beta_n-\alpha_n}{1-\beta_n}\right| \big\|T(\mu_n)y_n\big\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big[\|x_{n+1} - x_n\| - \varphi\big(\|x_{n+1} - x_n\|\big)\big] \\
&\quad + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\big(\|y_{n+1} - y_n\| + \big\|T(\mu_{n+1})y_n - T(\mu_n)y_n\big\|\big) \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|f(x_n)\big\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|T(\mu_n)y_n\big\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| + \big\|T(\mu_{n+1})y_n - T(\mu_n)y_n\big\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big(\big\|f(x_n)\big\| + \big\|T(\mu_n)y_n\big\|\big) \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\| + \sup_{y\in\{y_n\}}\big\|T(\mu_{n+1})y - T(\mu_n)y\big\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big(\big\|f(x_n)\big\| + \big\|T(\mu_n)y_n\big\|\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \sup_{y\in\{y_n\}}\big\|T(\mu_{n+1})y - T(\mu_n)y\big\| \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big(\big\|f(x_n)\big\| + \big\|T(\mu_n)y_n\big\|\big).
\end{aligned}
$$

It follows from the conditions (C1), (C2) and (C4), which implies that

$$
\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \leq 0.
$$

Applying Lemma 2.4, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and also

$$
\|x_{n+1} - x_n\| = (1-\beta_n)\|z_n - x_n\| \to 0
$$

as $n \to \infty$. Therefore, we have

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}
$$

Next, we show that $\lim_{n\to\infty}\|T(\mu_n)y_n - y_n\| = 0$. Since $p \in \mathcal{F}$, from Lemma 2.6, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n - p\|^2 \\
&\le \alpha_n \|f(x_n)-p\|^2 + (1-\alpha_n-\gamma_n)\|x_n-p\|^2 + \gamma_n\|y_n-p\|^2 \\
&= \alpha_n \|f(x_n)-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 - \gamma_n\big(\|x_n-p\|^2 - \|y_n-p\|^2\big) \\
&= \alpha_n \|f(x_n)-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 \\
&\quad - \gamma_n\big(\|x_n-p\| - \|y_n-p\|\big)\big(\|x_n-p\| + \|y_n-p\|\big) \\
&\le \alpha_n \|f(x_n)-p\|^2 + \|x_n-p\|^2 - \gamma_n\|x_n-y_n\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\gamma_n\|x_n-y_n\|^2 &\le \alpha_n \|f(x_n)-p\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\le \alpha_n \|f(x_n)-p\|^2 + \big(\|x_n-p\| + \|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|.
\end{aligned}$$

From the condition (C1) and (3.6), this implies that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Now, we note that

$$\begin{aligned}
\|x_n - T(\mu_n)y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(\mu_n)y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n - T(\mu_n)y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n\big(f(x_n) - T(\mu_n)y_n\big) + \beta_n\big(x_n - T(\mu_n)y_n\big)\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - T(\mu_n)y_n\| + \beta_n\|x_n - T(\mu_n)y_n\|.
\end{aligned}$$

Therefore, we get

$$\|x_n - T(\mu_n)y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - T(\mu_n)y_n\|.$$

From the conditions (C1), (C2) and (3.6), this implies that $\|x_n - T(\mu_n)y_n\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
\|x_n - T(\mu_n)x_n\| &\le \|x_n - T(\mu_n)y_n\| + \|T(\mu_n)y_n - T(\mu_n)x_n\| \\
&\le \|x_n - T(\mu_n)y_n\| + \|y_n - x_n\|,
\end{aligned}$$

and hence it follows that $\lim_{n\to\infty}\|T(\mu_n)x_n - x_n\| = 0$.

Next, we prove that $z \in \mathcal{F} := F(\mathcal{Q}) \cap F(\mathcal{S})$. By the reflexivity of E and boundedness of the sequence $\{x_n\}$, we may assume that $x_{n_i} \rightharpoonup z$ for some $z \in C$.

(a) First, we show that $z \in F(\mathcal{S})$. Put $x_i = x_{n_i}$, $\alpha_i = \alpha_{n_i}$, $\beta_i = \beta_{n_i}$, $\gamma_i = \gamma_{n_i}$ and $\mu_i = \mu_{n_i}$ for $i \in \mathbb{N}$, let $t_i \ge 0$ be such that

$$\mu_i \to 0 \quad \text{and} \quad \frac{\|T(\mu_i)x_i - x_i\|}{\mu_i} \to 0, \quad i \to \infty.$$

Fix $t > 0$. Notice that

$$\begin{aligned}\|x_i - T(t)p\| &\le \sum_{k=0}^{[t/\mu_i]-1} \|T((k+1)\mu_i)x_i - T(k\mu_i)x_i\| + \|T([t/\mu_i]\mu_i)x_i - T([t/\mu_i]\mu_i)z\| \\ &\quad + \|T([t/\mu_i]\mu_i)z - T(t)z\| \\ &\le [t/\mu_i]\|T(\mu_i)x_i - x_i\| + \|x_i - p\| + \|T(t - [t/\mu_i]\mu_i)z - z\| \\ &\le t\frac{\|T(\mu_i)x_i - x_i\|}{\mu_i} + \|x_i - p\| + \|T(t - [t/\mu_i]\mu_i)z - z\| \\ &\le t\frac{\|T(\mu_i)x_i - x_i\|}{\mu_i} + \|x_i - p\| + \max\{\|T(s)z - z\| : 0 \le s \le \mu_i\}.\end{aligned}$$

For all $i \in \mathbb{N}$, we have

$$\limsup_{i\to\infty}\|x_i - T(t)z\| \le \limsup_{i\to\infty}\|x_i - z\|.$$

Since the Banach space E with a weakly sequentially continuous duality mapping satisfies Opial's condition, this implies $T(t)z = z$. Therefore, $z \in F(\mathcal{S})$.

(b) Next, we show that $z \in F(\mathcal{Q})$. From Lemma 3.1, we know that $\mathcal{Q} = \mathcal{Q}_C^M$ is nonexpansive; it follows that

$$\|y_n - \mathcal{Q}y_n\| = \|\mathcal{Q}_C^M x_n - \mathcal{Q}_C^M y_n\| \le \|x_n - y_n\|.$$

Thus $\lim_{n\to\infty}\|y_n - \mathcal{Q}y_n\| = 0$. Since $\mathcal{Q}$ is nonexpansive, we get

$$\begin{aligned}\|x_n - \mathcal{Q}x_n\| &\le \|x_n - y_n\| + \|y_n - \mathcal{Q}y_n\| + \|\mathcal{Q}y_n - \mathcal{Q}x_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - \mathcal{Q}y_n\|,\end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n - \mathcal{Q}x_n\| = 0. \tag{3.7}$$

By Lemma 2.7 and (3.7), we have $z \in F(\mathcal{Q})$. Therefore, $z \in \mathcal{F}$.

Next, we show that $\limsup_{n\to\infty}\langle(f - I)\bar{x}_1, J(x_n - \bar{x}_1)\rangle \le 0$, where $\bar{x}_1 = Q_{\mathcal{F}}f(\bar{x}_1)$. Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ where $x_{n_i} \rightharpoonup z$ such that

$$\limsup_{n\to\infty}\langle(f - I)\bar{x}_1, J(x_n - \bar{x}_1)\rangle = \lim_{i\to\infty}\langle(f - I)\bar{x}_1, J(x_{n_i} - \bar{x}_1)\rangle. \tag{3.8}$$

Now, from (3.8), Proposition 2.1(iii) and the weakly sequential continuity of the duality mapping J, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(f - I)\bar{x}_1, J(x_n - \bar{x}_1)\rangle &= \lim_{i\to\infty}\langle(f - I)\bar{x}_1, J(x_{n_i} - \bar{x}_1)\rangle \\ &= \langle(f - I)\bar{x}_1, J(z - \bar{x}_1)\rangle \le 0.\end{aligned} \tag{3.9}$$

From (3.6), it follows that

$$\limsup_{n\to\infty}\big\langle (f-I)\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \le 0. \tag{3.10}$$

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}}f(\bar{x}_1)$. We compute that

$$\begin{aligned}
\|x_{n+1}-\bar{x}_1\|^2 &= \big\langle x_{n+1}-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&= \big\langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n - \bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&= \big\langle \alpha_n\big(f(x_n)-\bar{x}_1\big) + \beta_n(x_n-\bar{x}_1) + \gamma_n\big(T(\mu_n)y_n-\bar{x}_1\big), J(x_{n+1}-\bar{x}_1)\big\rangle \\
&= \alpha_n\big\langle f(x_n)-f(\bar{x}_1), J(x_{n+1}-\bar{x}_1)\big\rangle + \alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&\quad + \beta_n\big\langle x_n-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle + \gamma_n\big\langle T(\mu_n)y_n-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&\le \alpha_n\big[\|x_n-\bar{x}_1\| - \varphi\big(\|x_n-\bar{x}_1\|\big)\big]\|x_{n+1}-\bar{x}_1\| + \alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&\quad + \beta_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| + \gamma_n\|y_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\le \alpha_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| - \alpha_n\varphi\big(\|x_n-\bar{x}_1\|\big)\|x_{n+1}-\bar{x}_1\| \\
&\quad + \alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&\quad + \beta_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| + \gamma_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&= \|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| - \alpha_n\varphi\big(\|x_n-\bar{x}_1\|\big)\|x_{n+1}-\bar{x}_1\| \\
&\quad + \alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle \\
&= \frac{1}{2}\big(\|x_n-\bar{x}_1\|^2 + \|x_{n+1}-\bar{x}_1\|^2\big) - \alpha_n\varphi\big(\|x_n-\bar{x}_1\|\big)\|x_{n+1}-\bar{x}_1\| \\
&\quad + \alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle.
\end{aligned}$$

By (3.5) and since $\{x_{n+1}-\bar{x}_1\}$ is bounded, *i.e.*, there exists $M > 0$ such that $\|x_{n+1}-\bar{x}_1\| \le M$, which implies that

$$\|x_{n+1}-\bar{x}_1\|^2 \le \|x_n-\bar{x}_1\|^2 - 2\alpha_n M\varphi\big(\|x_n-\bar{x}_1\|\big) + 2\alpha_n\big\langle f(\bar{x}_1)-\bar{x}_1, J(x_{n+1}-\bar{x}_1)\big\rangle. \tag{3.11}$$

Now, from (C1) and applying Lemma 2.5 to (3.11), we get $\|x_n-\bar{x}_1\| \to 0$ as $n\to\infty$. This completes the proof. □

Corollary 3.4 *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \ge 0\}$ be a nonexpansive semigroup on C and Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be a β-inverse-strongly accretive such that $\beta \ge \lambda K^2$ where K is the best smooth constant. Let f be a weakly contractive mapping of C into itself with function φ. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be in $(0,1)$ with $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1$, $n \ge 1$, $\{\mu_n\} \subset (0,\infty)$ and satisfy the conditions (C1)-(C4) in Theorem 3.3. Suppose $\mathcal{F} := F(\mathcal{Q}) \cap F(\mathcal{S}) \ne \emptyset$, where $\mathcal{Q}$ is defined by*

$$\mathcal{Q}(x) = Q_C(I-\lambda A)Q_C(I-\lambda A)\cdots Q_C(I-\lambda A)x, \quad \forall x \in C,$$

and λ be a positive real number. For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ are generated by

$$\begin{cases} y_n = Q_C(I - \lambda A)Q_C(I - \lambda A)\cdots Q_C(I - \lambda A)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n. \end{cases} \tag{3.12}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof Putting $A = A_M = A_{M-1} = \cdots = A_2 = A_1$, $\beta = \beta_M = \beta_{M-1} = \cdots = \beta_2 = \beta_1$ and $\lambda = \lambda_M = \lambda_{M-1} = \cdots = \lambda_2 = \lambda_1$ in Theorem 3.3, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.5 *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C and Q_C be a sunny nonexpansive retraction from E onto C. Let $A_l : C \to E$ be a β_l-inverse-strongly accretive such that $\beta_l \geq \lambda_l K^2$, where $l \in \{1, 2\}$ and K be the best smooth constant. Let f be a weakly contractive mapping of C into itself with function φ. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be in $(0, 1)$ with $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1$, $n \geq 1$, $\{\mu_n\} \subset (0, \infty)$ and satisfy the conditions (C1)-(C4) in Theorem 3.3. Suppose $\mathcal{F} := F(\mathcal{Q}) \cap F(\mathcal{S}) \neq \emptyset$, where $\mathcal{Q}$ is defined by*

$$\mathcal{Q}(x) = Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x, \quad \forall x \in C,$$

and λ_1, λ_2 are positive real numbers. For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ are generated by

$$\begin{cases} y_n = Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n. \end{cases} \tag{3.13}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$ and $(\bar{x}_1, \bar{x}_2)$ is a solution of the problem (1.6), where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof Taking $M = 2$ in Theorem 3.3, we can conclude the desired conclusion easily. This completes the proof. □

4 Some applications

4.1 (I) Application to strictly pseudocontractive mappings

Let E be a Banach space and let C be a subset of E. Recall that a mapping $T : C \to C$ is said to be *k-strictly pseudocontractive* if there exist $k \in [0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\left\|(I - T)x - (I - T)y\right\|^2 \tag{4.1}$$

for all $x, y \in C$. Then (4.1) can be written in the following form:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \frac{1-k}{2}\left\|(I - T)x - (I - T)y\right\|^2. \tag{4.2}$$

Moreover, we know that A is $\frac{1-k}{2}$-inverse strongly monotone and $A^{-1}0 = F(T)$ (see also [9]).

Theorem 4.1 *Let E be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C and $T_l : C \to C$ be a k_l-strictly pseudocontractive mapping with $\lambda_l \leq \frac{(1-k_l)}{2K^2}$, $l \in \{1, 2, \dots, M\}$. Let f be a weakly contractive mapping of C into itself with function φ and suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1$, $n \geq 1$ and $\{\mu_n\} \subset (0, \infty)$. Suppose $\mathcal{F} := F(\mathcal{S}) \cap (\bigcap_{l=1}^{M} F(T_l)) \neq \emptyset$ and let λ_l, $l = 1, 2, \dots, M$ be positive real numbers. If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iii) $\lim_{n\to\infty} \mu_n = 0$*;*

(iv) $\lim_{n\to\infty} \sup_{y\in\tilde{C}} \|T(\mu_{n+1})y - T(\mu_n)y\| = 0$, *$\tilde{C}$ bounded subset of C.*

Then the sequence $\{x_n\}$ is generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = ((1-\lambda_M) + \lambda_M T_M)((1-\lambda_{M-1}) + \lambda_{M-1} T_{M-1}) \cdots ((1-\lambda_2) + \lambda_2 T_2) \\ \qquad \times ((1-\lambda_1) + \lambda_1 T_1) x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) y_n \end{cases} \tag{4.3}$$

converges strongly to $Q_{\mathcal{F}}$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of E onto $\mathcal{F}$.

Proof Putting $A_l = I - T_l$, $l \in \{1, 2, \dots, M\}$. From (4.2), we get A_l is $\frac{1-k_l}{2}$-inverse strongly accretive operator. It follows that $\mathrm{GSVI}(C, A_l) = \mathrm{GSVI}(C, I - T_l) = F(T_l) \neq \emptyset$ and $(\bigcap_{l=1}^{M} \mathrm{GSVI}(C, I - T_l)) = F(Q) \Leftrightarrow$ is the solution of the problem (1.7) (see also Ceng *et al.* [3, Theorem 4.1, pp.388-389] and Aoyama *et al.* [1, Theorem 4.1, p.10]).

$$\begin{aligned} &\big((1-\lambda_1) + \lambda_1 T_1\big) x_n = Q_C\big((1-\lambda_1) + \lambda_1 T_1\big) x_n \\ &\vdots \\ &\big((1-\lambda_M) + \lambda_M T_M\big) \cdots \big((1-\lambda_1) + \lambda_1 T_1\big) x_n \\ &\quad = Q_C\big((1-\lambda_M) + \lambda_M T_M\big) \cdots Q_C\big((1-\lambda_1) + \lambda_1 T_1\big) x_n. \end{aligned}$$

Therefore, by Theorem 3.3, $\{x_n\}$ converges strongly to some element $\bar{x}_1$ of $\mathcal{F}$. □

4.2 (II) Application to Hilbert spaces

In real Hilbert spaces H, by Lemma 3.2, we obtain the following lemma:

Lemma 4.2 *For given $(x_1^*, x_2^*, \dots, x_M^*)$, a solution of the problem is as follows:*

$$\begin{cases} \langle \lambda_M A_M x_M^* + x_1^* - x_M^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, x - x_M^* \rangle \geq 0, \quad \forall x \in C, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{4.4}$$

if and only if

$$x_1^* = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*$$

is a fixed point of the mapping $\mathcal{P} : C \to C$ defined by

$$\mathcal{P}(x) = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C,$$

where P_C is a metric projection H onto C.

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem 3.3, we can obtain the following result immediately.

Theorem 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_l : C \to H$ be a β_l-inverse-strongly monotone mapping with $\lambda_l \in (0, 2\beta_l)$, $l \in \{1, 2, \ldots, M\}$. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C and f be a weakly contractive mapping of C into itself with function φ. Assume that $\mathcal{F} := F(\mathcal{P}) \cap F(S) \neq \emptyset$, where $\mathcal{P}$ is defined by Lemma 4.2 and let λ_l, $l = 1, 2, \ldots, M$ be positive real numbers. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ with $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1$, $n \geq 1$ and the following conditions be satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lim_{n\to\infty} \mu_n = 0$;
(iv) $\lim_{n\to\infty} \sup_{y\in\tilde{C}} \|T(\mu_{n+1})y - T(\mu_n)y\| = 0$, *$\tilde{C}$ bounded subset of C.*

For arbitrary given $x_0 = x \in C$, the sequences $\{x_n\}$ are generated by

$$\begin{cases} y_n = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n. \end{cases} \tag{4.5}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = P_{\mathcal{F}} f(\bar{x}_1)$ and $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_M)$ is a solution of the problem (4.4).

Remark 4.4 We can replace a contraction mapping f to a weak contractive mapping by setting $\varphi(t) = (1 - k)t$. Hence, our results can be obtained immediately.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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